Heuristic proof of the Theorem of Borsuk and Ulam

Theo W. Ruijgrok

Institute of Theoretical Physics Utrecht University, Utrecht, The Netherlands

Abstract

We give a heuristic proof of the statement that there exist two antipodal points at which the atmospheric pressures and temperatures are equal.

1. Statement of the theorem

Suppose that at each point x on the surface of Earth, the temperature $T(x)$ and the pressure $P(x)$ are measured. They are supposed to be continuous functions of x. Let the mapping of x into its antipode x' be denoted by A, so that $x' = Ax$.

The theorem of Borsuk and Ulam now asserts the following:
There exists a point x_0 with antipode $x_0' = Ax_0$, for which $T(x_0) = T(x_0')$ and $P(x_0) = P(x_0')$.

Before turning to the proof the theorem is reformulated by first defining the difference functions $F_T(x) = T(x) - T(Ax)$ and $F_P(x) = P(x) - P(Ax)$. These functions are antisymmetric under A, *i.e.*, $F_T(Ax) = -F_T(x)$ and $F_P(Ax) = -F_P(x)$. For the theorem to be true it must now be shown that there exists a pair of antipodes (x_0, x_0') for which $F_T(x_0) = 0$ and $F_P(x_0) = 0$.

2. Proof

Consider an arbitrary real and continuous function $F(x)$ defined on the surface of Earth, with the property that $F(Ax) = -F(x)$. The regions where $F(x) > 0$ will be called *islands*; the regions where $F(x) < 0$ are called *lakes* and the lines where $F(x) = 0$ are *shorelines*. There may be lakes on an island and there may be islands in a lake, but the total land area is equal to the total area covered by water. By definition shorelines cannot intersect, although they can touch each other.

It is clear that for every shoreline s there is an antipodal shoreline $s' = As$, which has no intersection with s. In addition there may be one or more singly connected shorelines which are antipodal to itself. Such a line will be called *equator*. This name may be confusing, because it can have a very irregular and even fractal form. It shares, however, one common property with the Equator where

Neptune rules, in that it divides the surface of the earth into two hemispheres of equal area. This property makes it impossible for two equators to exist, since they necessarily would have to intersect each other, which is forbidden for shorelines. The remaining and essential part of the proof consists in showing that there is always one equator.

For that purpose consider an adventurous traveller, who is living at a point x_s, which is situated on an island. He owns an amphibious vehicle, with which he wants to make a trip to his brother, who is living on a yacht, anchored at the antipodal position $x_s' = Ax_s$, in a lake of the same form as his brother's island.

It can happen that the second shore of the lake around the traveller's island completely encircles his place. This means that on the other side there must also be a second antipodal shore line completely surrounding the point x_s'. Therefore, if there were no equator, the traveller could reach his brother only by crossing an even number of shore lines, after which he finds himself on an island again. The brother, however, lives in the middle of a lake. This leads to a contradiction and consequently there must be an equator. This equator has on one side a continent, which encircles the earth, with its antipodal ocean on the other side.

The situation is illustrated by considering as an example the following function for the temperature

$$T(\theta, \varphi) = 0.2\,\theta^2 \cos\theta + \cos(17\,\theta) + 0.3\cos(19\,\varphi) + \sin\varphi + 2\sin(11\,\theta)\cos(7\,\varphi).$$

The height of the difference $F_T(\theta, \varphi) = T(\theta, \varphi) - T(\pi - \theta, \varphi + \pi)$ is plotted in figure 1.

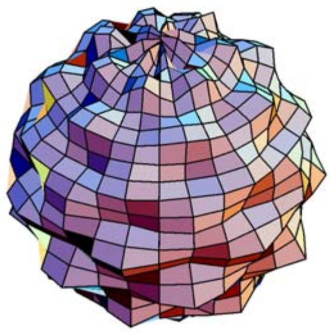

Figure 1. Plot of the temperature difference $T(\theta, \varphi) - T(\pi - \theta, \varphi + \pi)$

The regions where this function is positive and negative are shown in figure 2, which has some similarity with the art of Escher. The third picture gives the shore lines and shows clearly the occurrence of a rather rugged equator, separating a Northern from a Southern hemisphere.

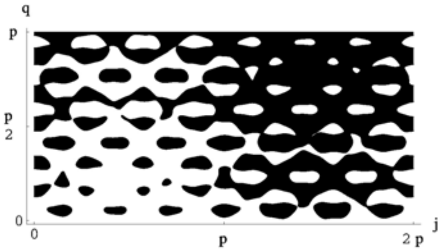

Figure 2. Domains of positive and negative temperature differences

A complication arises when there exist patches B together with their antipodal patches AB, for which $F(B) = 0$ and $F(AB) = 0$. These will be called *beaches*. It is believed that the occurrence of such beaches will not invalidate the proof of the existence of a single equator.

The proof of the Borsuk–Ulam theorem is concluded by considering the two equators belonging to the functions $F_T(x)$ and $F_P(x)$. These equators must intersect in a pair of antipodal points (x_0, x_0'), because each of them is singly connected and divides the surface of the sphere into two equal parts. These are then the points where the temperatures and pressures are equal.

It would be interesting if meteorologists could demonstrate the existence of an "equatorial" band, for which the atmospheric pressure in each point of the band turned out to be larger than in its antipodal point. Although this band may have a very strange form and vary from one moment to the other, its existence is guaranteed by the theorem of Borsuk and Ulam.

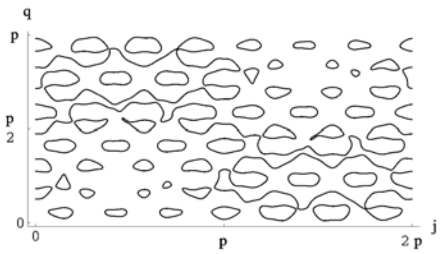

Figure 3. Shore lines and equator

The extensive literature on the Borsuk–Ulam theorem can be traced from the book by Matoušek [1]. However, in spite of this abundance, the author has not yet been able to show that the present proof is a disguised, but equivalent form of any of the many existing proofs.

[1] J. Matoušek, *Using the Borsuk–Ulam Theorem*, Springer, Berlin, 2003.